

## THE PLATEAU PROBLEM FOR SURFACES OF PRESCRIBED MEAN CURVATURE IN A RIEMANNIAN MANIFOLD

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### 1. Introduction

In this work we treat the problem of finding a surface of prescribed mean curvature in a three-dimensional riemannian manifold  $M$ , with a given closed curve as boundary. That is, given a real-valued function  $H(z)$  defined on  $M$ , we wish to find a mapping  $z: B \rightarrow M$ ,  $B$  denoting the two-dimensional unit disk, which satisfies the following conditions:

- (i)  $z \in C^2(B) \cap C^0(\bar{B})$ ,
- (ii)  $z$  maps  $\partial B$  homeomorphically onto  $\Gamma$ ,
- (iii)  $z$  satisfies in  $B$  the systems

$$(1.1) \quad \nabla_{z_u} z_u + \nabla_{z_v} z_v = 2H(z)^*(z_u \wedge z_v),$$

$$(1.2) \quad \langle z_u, z_u \rangle - \langle z_v, z_v \rangle = \langle z_u, z_v \rangle = 0.$$

Here  $\langle \cdot, \cdot \rangle$  denotes the inner product on the tangent bundle of  $M$ ,  $\nabla$  the associated Levi-Civita connection,  $*P$  the tangent vector associated with a two-vector  $P$  using  $\langle \cdot, \cdot \rangle$ . Let  $g_{ij}$  be the coefficients of  $\langle \cdot, \cdot \rangle$  in some coordinate system. We may write explicitly

$$*(z_u \wedge z_v)^k = \sqrt{g} \begin{vmatrix} g^{1k} & g^{2k} & g^{3k} \\ z_u^1 & z_u^2 & z_u^3 \\ z_v^1 & z_v^2 & z_v^3 \end{vmatrix},$$

where  $g_{ij}g^{jk} = \delta_i^k$  and  $g = \det(g_{ij})$ . (1.2) states that  $z$  is a conformal mapping on its image (possibly with degenerate points); under that condition, (1.1) become the equations for mean curvature  $H(z)$  at regular points.

The basic result of the present paper for smooth complete  $M$  may be stated as follows. Let  $K_0$  denote an upper bound on sectional curvatures of  $M$ , and  $\Phi(r)$  the mean curvature with respect to an inward normal of the geodesic sphere of radius  $r$  in the space of constant curvature  $K_0$ . Explicitly,  $\Phi(r) = \sqrt{K_0} \cot(\sqrt{K_0} r)$ . In the case  $K_0 > 0$ , replace  $\Phi$  by any smaller function  $\phi$

which is monotone decreasing. Then for a rectifiable Jordan curve  $\Gamma$  contained in the geodesic ball  $B_r(m)$ , where  $\exp_m$  is injective on  $B_r(0) \subset M_m$ , and for a Hölder-continuous function  $H(z)$  satisfying  $|H(z)| \leq \phi(r)$  in that ball, the problem has a solution (Theorem 2). The injectivity of  $\exp_m$  on  $B_r(0)$  is not essential (Theorem 3).

For the minimal surface case, i.e.,  $H \equiv 0$ , this problem was considered by Morrey [14] after pioneering work in euclidean space by Radó [16] and Douglas [2]. Heinz [7] considered the case of constant mean curvature  $H$  in euclidean space, showing existence under the condition that  $\Gamma$  be contained in a ball of radius  $(\sqrt{17} - 1)/(8|H|)$ . This radius was sharpened by Werner [19] to  $\frac{1}{2}|H|^{-1}$ . Hildebrandt [11] improved this to the best possible, requiring radius  $|H|^{-1}$ . This was accomplished, using regularity results of Morrey, via the introduction of a restricted variational problem in combination with a new maximum principle valid for solutions which are only continuous. Hildebrandt has generalized this result to prescribed mean curvature using a more elegant proof which involves a modified free variational problem [10]. This method appears to run into difficulty in the riemannian context if positive sectional curvatures are allowed. However a variant of the method of [11] is applied to the problem successfully in the present work: the result stated above is a direct generalization of the result of [10]. In the case of nonpositive sectional curvatures our method provides a generalization of Morrey's result in [14]. As in [11], the core of this work is a maximum principle for continuous solutions to the variational problem, which we present in § 4. A similar maximum principle, requiring the mapping to be smooth, has been recently obtained by Kaul [12].

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## 2. The functional and its first variation

In order to define the variational problem we shall use, we assume for some point  $m \in M$  the map  $\exp = \exp_m: M_m \rightarrow M$  is a diffeomorphism of the ball  $B_R(0)$  onto its image, which is the geodesic ball  $B_R$  of radius  $R$  and center  $m$ . If  $S$  is a set in  $M_m$  we define  $C(S)$ , the cone on  $S$ , to be the set  $\{tx: 0 \leq t \leq 1, x \in S\}$ . If  $S_1 = \exp(S) \subset B_R$  we define the geodesic cone on  $S_1$ ,  $C(S_1) = \exp(C(S))$ . If  $S_1$  is an oriented 2-chain,  $C(S_1)$  is an oriented 3-chain. Let a mapping  $z: B \rightarrow B_R$  and a measurable real-valued function  $H(x)$  defined on  $B_R$  be given. Then we may define the functional

$$W[z] = 4 \int_{C(z(B))} H(x) dV(x),$$

where  $dV$  refers to oriented riemannian volume. For vectors  $V_1, V_2$  tangent to  $M_m$  we write the euclidean inner product as  $V_1 \cdot V_2$ , reserving the symbol  $\langle V_1, V_2 \rangle$  for the riemannian inner product on tangent vectors to  $M$ . For

$z: B \rightarrow M$  and  $y: B \rightarrow M_m$  we use the notation  $|\nabla y|^2 = y_u \cdot y_u + y_v \cdot y_v$  and  $|\nabla z|_M^2 = \langle z_u, z_u \rangle + \langle z_v, z_v \rangle$ . We may then write the euclidean and riemannian Dirichlet integrals as

$$\bar{D}[y] = \iint_B |\nabla y|^2 dudv, \quad D[z] = \iint_B |\nabla z|_M^2 dudv.$$

Finally define the functional for our variational problem:

$$E[z] = D[z] + W[z].$$

For a mapping  $z$  into  $B_R$  we introduce the notation  $\tilde{z} = \exp^{-1} \circ z$ . The manifold  $M$  is said to be of class  $C^k$  if it has a  $C^k$  differentiable structure and the inner product  $\langle \cdot, \cdot \rangle$  is of class  $C^{k-1}$ .

**Proposition 1.** *If  $M$  is of class  $C^2$  and  $H \in C^1$ , then the Euler equations for  $E$  are the system (1.1).*

*Proof.* We may write

$$\begin{aligned} W[z] &= 4 \iint_B \int_0^1 H(tz) \langle \gamma_*(t|\tilde{z}|), *(z_u(t|\tilde{z}|) \wedge z_v(t|\tilde{z}|)) \rangle |\tilde{z}| dt dudv \\ &= 4 \iint_B \omega(z, \nabla z) dudv. \end{aligned}$$

Here we define  $tz = \exp(t\tilde{z})$ ;  $\gamma$  is the arc-length geodesic from  $m$  to  $z$ ; for a vector  $V \in M_z$ ,  $V(t|\tilde{z}|)$  denotes the element at  $tz = \gamma(t|\tilde{z}|)$  of the Jacobi field along  $\gamma$  determined by  $V(|\tilde{z}|) = V$  and  $V(0) = 0$ . Let  $g_{ij}$  be the coefficients of the inner product of  $M$  with respect to normal coordinates at  $m$ ,  $g = \det(g_{ij})$ . Then

$$\begin{aligned} \omega(z, \nabla z) &= \int_0^1 H(tz) \sqrt{g}(tz) \frac{\tilde{z}}{|\tilde{z}|} \cdot (t\tilde{z}_u) \wedge (t\tilde{z}_v) |\tilde{z}| dt \\ (2.1) \quad &= \int_0^1 t^2 H(tz) \sqrt{g}(tz) dt \tilde{z} \cdot \tilde{z}_u \wedge \tilde{z}_v = Q(\tilde{z}) \cdot \tilde{z}_u \wedge \tilde{z}_v. \end{aligned}$$

The hypotheses imply  $Q \in C^1$ . By Lemma 8 of [5] the first variation (in the euclidean context)

$$[\omega]_z = \operatorname{div} Q\tilde{z}_u \wedge \tilde{z}_v,$$

where the divergence operator is euclidean. But  $\operatorname{div} Q = H(z) \sqrt{g}(z)$ , by way of integration by parts. The first variation (again in the euclidean context) of  $D$  in the  $l$ th component  $z^l$  is readily calculated as

$$-2g_{ki}(z_{uu}^k + z_{vv}^k) - 2\Gamma_{ij|l}(z_u^i z_u^j + z_v^i z_v^j) .$$

Here the  $\Gamma$ 's are the coefficients of  $\mathcal{V}$ . Thus the Euler equations of  $E$  are

$$g_{ki}[z_{uu}^k + z_{vv}^k + \Gamma_{ij}^k(z_u^i z_u^j + z_v^i z_v^j)] = 2H(z)\sqrt{g}(\tilde{z}_u \wedge \tilde{z}_v)^i ,$$

or

$$(2.2) \quad z_{uu}^k + z_{vv}^k + \Gamma_{ij}^k(z_u^i z_u^j + z_v^i z_v^j) = 2H(z)^*(z_u \wedge z_v)^k ,$$

which is the same as

$$\mathcal{V}_{z_u} z_u + \mathcal{V}_{z_v} z_v = 2H(z)^*(z_u \wedge z_v) .$$

### 3. Variation with fixed boundary mapping

Let  $M$  be a three-dimensional riemannian manifold of class  $C^3$ , with sectional curvatures  $K \leq b^2$ ,  $b$  either real or imaginary. Choose  $r_0 > 0$ ; if  $b^2 > 0$  we require  $|b|r_0 \leq \frac{1}{2}\pi$ . We assume for the present that  $\exp$  is defined on  $B_{r_0}(0) \subset M_m$  and is a diffeomorphism of  $B_{r_0}(0)$  onto  $B_{r_0}$  (this requirement will be dropped in § 6). The radius  $r_0$  will be fixed in this section and the one following.

Define  $\mathcal{D}_R = \{x: B \rightarrow M_m: x \in H_1(B), |x(w)| \leq R \text{ for almost all } w \in B\}$ . Here  $H_1$  denotes the space of  $L^2$  functions with  $L^2$  first derivatives and norm  $\|x\|_1$  given by  $\|x\|_1^2 = \|x\|_{L^2}^2 + \bar{D}[x]$ . For  $f \in \mathcal{D}_R$ , we define  $\mathcal{D}_R(f) = \{x \in \mathcal{D}_R: x - f \in \dot{H}_1(B)\}$ , where  $\dot{H}_1$  denotes the closure in  $H_1$  of smooth functions with compact support. For a given function  $H \in L^\infty(M)$ , we denote by  $P(f, R, H)$  the variational problem:  $E[y] \rightarrow \min$  among mappings  $y$  such that  $\tilde{y} \in \mathcal{D}_R(f)$  and by  $e(f, R, H)$  this minimum. Denote  $h = \text{ess sup}_{x \in B_{r_0}} |H(x)|$ .

Consider a mapping  $z: B \rightarrow B_{r_0}$  at a point  $w_0 \in B$ . We need to bound  $\omega(z, \mathcal{V}z)$  in terms of the riemannian Dirichlet integrand  $|\mathcal{V}z|_M^2$ .

Define a function on  $C$ :

$$G(\zeta) = \csc^2 \zeta \cot \zeta (\zeta - \cos \zeta \sin \zeta) \quad \zeta \neq 0 ,$$

$$G(0) = 2/3 .$$

Observe  $G$  is continuous, with  $0 \leq G(\zeta) < 1$  for  $-\frac{1}{2}\pi \leq \zeta \leq \frac{1}{2}\pi$  and for all imaginary  $\zeta$ .

**Lemma 1.** *If  $r = |z(w_0)| \leq r_0$  and  $h \leq b \cot(br_0)$ , then*

$$|\omega(z(w_0), \mathcal{V}z(w_0))| \leq \frac{1}{4} |\mathcal{V}z|_M^2 G(br_0) .$$

*Proof.* Let  $F(\rho) = \sqrt{g}(\rho z_0/r)$  be the Jacobian of  $\exp$  at  $\rho z_0/r \in M_m$ . Here  $z_0 = z(w_0)$ . Then from (2.1) we have

$$\omega(z_0, \nabla z(w_0)) = \int_0^1 t^2 H(tz_0) F(tr) dt \tilde{z}_0 \cdot \tilde{z}_u(w_0) \wedge \tilde{z}_v(w_0) .$$

Writing  $A = \tilde{z}_0 \cdot \tilde{z}_u(w_0) \wedge \tilde{z}_v(w_0)$ , note that

$$|AF(r)| = r | \langle \gamma_*, *(z_u(w_0) \wedge z_v(w_0)) \rangle | \leq \frac{1}{2} r |\nabla z(w_0)|_M^2 ,$$

where  $\gamma$  is the arc-length geodesic from  $m$  to  $z_0$ . Now  $F$  satisfies the growth condition

$$\frac{\rho^2 F(\rho)}{r^2 F(r)} \leq \frac{\sin^2(b\rho)}{\sin^2(br)}$$

for  $\rho \leq r$  [6]. So we have

$$\omega(z_0, \nabla z(w_0)) = A \int_0^1 t^2 H(tz_0) F(tr) dt \leq AF(r) h \int_0^1 \frac{t^2 F(tr)}{F(r)} dt ,$$

so that

$$\begin{aligned} |\omega(z_0, \nabla z(w_0))| &\leq \frac{1}{2} r |\nabla z(w_0)|_M^2 h \int_0^1 \frac{\sin^2(btr)}{\sin^2(br)} dt \\ &= \frac{1}{2} |\nabla z(w_0)|_M^2 h \int_0^r \frac{\sin^2(b\rho)}{\sin^2(br)} d\rho . \end{aligned}$$

The integral is increasing as a function of  $r$  since

$$\frac{d}{dr} \int_0^r \frac{\sin^2(b\rho)}{\sin^2(br)} d\rho = 1 - G(br) > 0 .$$

Thus

$$\begin{aligned} |\omega(z_0, \nabla z(w_0))| &\leq \frac{1}{2} |\nabla z(w_0)|_M^2 b \cot(br_0) \int_0^{r_0} \frac{\sin^2(b\rho)}{\sin^2(br_0)} d\rho \\ &= \frac{1}{4} |\nabla z(w_0)|_M^2 G(br_0) . \quad \text{q.e.d.} \end{aligned}$$

In the case  $b = 0$ , read  $r$  for  $\sin(br)/b$  and 1 for  $\cos(br)$ . Thus  $\Phi(r) = b \cot(br)$  becomes the familiar  $1/r$  for  $b = 0$ .

For a vector  $V$  tangent to  $B_{r_0}$  denote  $\tilde{V} = (\exp^{-1})_*(V)$ .

**Lemma 2.** Assume  $M$  is of class  $C^1$ . There exists  $N$  such that for any tangent vectors  $V \in M_x, |\tilde{Z}| \leq r_0$ , we have

$$\frac{1}{N} \tilde{V} \cdot \tilde{V} \leq \langle V, V \rangle \leq N \tilde{V} \cdot \tilde{V} .$$

*Proof.* At each point  $z$  of  $\bar{B}_{r_0}$ , let  $N(z) = \sup \{ \langle V, V \rangle, 1/\langle V, V \rangle : V \in M_z, \tilde{V} \cdot \tilde{V} = 1 \}$ . Since  $\langle, \rangle$  is continuous and positive definite,  $N(z)$  is continuous and finite, and hence bounded on  $\bar{B}_{r_0}$ .

**Corollary 1.** *If  $h \leq b \cot (br_0)$  and  $\tilde{z} \in \mathcal{D}_{r_0}$ , then for any measurable  $B' \subset B$  we have*

$$\frac{1}{N} [1 - G(br_0)] \bar{D}_{B'}[\tilde{z}] \leq E_{B'}[z] \leq N [1 + G(br_0)] \bar{D}_{B'}[\tilde{z}] .$$

Here the subscripted  $B'$  denotes integration over that set.

**Lemma 3.** *Assume  $h \leq b \cot (br_0)$ . Let  $\{\tilde{y}_n\}$  be a sequence from  $\mathcal{D}_R$ ,  $R \leq r_0$ , such that  $\tilde{y}_n$  converges weakly to  $\tilde{y}$  in  $H_1(B)$ . Then  $\tilde{y} \in \mathcal{D}_R$  and  $E[y] \leq \liminf E[y_n]$ .*

*Proof.* Using Lemma 1, as in [11, Lemma 1].

**Lemma 4.** *Suppose  $h \leq b \cot (br_0)$  and  $f \in \mathcal{D}_R$  for  $R \leq r_0$ . Then there exists a solution  $z$  to the variational problem  $P(f, R, H)$ .*

*Proof.* Choose a sequence  $\tilde{z}_n \in \mathcal{D}_R(f)$  such that, with  $z_n = \exp \circ \tilde{z}_n$ ,  $\lim E[z_n] = e(f, R, H)$ . Then the numbers  $E[z_n]$  are uniformly bounded, hence by Corollary 1,  $\bar{D}[\tilde{z}_n] <$  uniform bound. So  $\|\tilde{z}_n\|_1^2 \leq R^2 \iint_B dudv + \bar{D}[\tilde{z}_n] <$  uniform bound, and some subsequence converges weakly to a function  $\tilde{z} \in H_1(B)$ , with  $\tilde{z} - f \in \dot{H}_1(B)$ . Using Lemma 3,  $\tilde{z} \in \mathcal{D}_R(f)$  and

$$e(f, R, H) \leq E[z] \leq \lim E[z_n] = e(f, R, H) .$$

Thus  $z$  solves  $P(f, R, H)$ . q.e.d.

As a consequence of its minimizing property and Corollary 1, this  $z$  satisfies a uniform Hölder condition in  $B$ . If, moreover,  $f \in C^0(\partial B)$  then  $z \in C^0(\bar{B})$  and  $z = f$  on  $B$ . These properties follow from results of Morrey [13, Theorem 2.2] using a glueing technique (cf. [11, Lemma 4]). For any subdomain  $B' \subset B$  such that  $\sup_{w \in \bar{B}'} |\tilde{z}(w)| < R$ , the first variation of  $E$  will vanish with respect to any smooth test function with compact support; that is, for  $H \in C^1$ ,  $z$  is a weak solution to the Euler equations (1.1) or the equivalent form (2.2). It then follows from a result of Heinz and Tomi (cf. [18]) that  $z \in C^{1+\beta}$  for all  $\beta < 1$  and has the representation

$$(3.1) \quad z(w) = y(w) + \iint_{B'} G(w, \zeta) \{ 2H(z(\zeta))^* (z_\xi \wedge z_\eta) - \Gamma_{ij}^k (z_\xi^i z_\zeta^j + z_\eta^i z_\zeta^j) V_k \} d\xi d\eta ,$$

where  $y$  is the harmonic function with  $z = y$  on  $\partial B'$ ,  $G(w, \zeta)$  the Green's function for  $B'$ , and  $V_k$  the  $k$ th coordinate vector. Assuming only that  $H$  is  $C^\alpha$ , it follows by methods of potential theory that  $z \in C^{2+\alpha}$  and satisfies (1.1). It is

the purpose of the next section to show that under appropriate hypotheses these considerations may be applied with  $B' = B$  itself.

**4. The maximum principle; smoothness**

Let  $r_1 \in (0, r_0)$  be chosen. We construct a  $C^1$  mapping  $\tilde{T}: M_m \rightarrow M_m$  by defining  $\tilde{T}(y) = \sigma(|y|)y/|y|$  for a  $C^1$  function  $\sigma$  with the properties:  $\sigma(r) \leq r$  for all  $r$ ,  $\sigma(r) = r$  for  $r \in [0, r_1]$ , and  $\sigma''(r_1+) < 0$ . Now define  $T: B_{r_0} \rightarrow B_{r_0}$  by  $T(x) = \exp(\tilde{T}(\tilde{x}))$ . Observe that if  $y \in \mathcal{D}_R$  then  $\tilde{T} \circ y \in \mathcal{D}_{\sigma(R)}$ .

**Lemma 5.** *Suppose  $h < b \cot(br_1)$ . Then there exists  $R_1, r_1 < R_1 \leq r_0$ , such that for  $\tilde{z} \in \mathcal{D}_{R_1} \cap C^0(\bar{B})$  with  $\inf_{w \in B} |\tilde{z}(w)| \leq r_1 < \sup_{w \in B} |\tilde{z}(w)|$  there holds  $E[T \circ z] < E[z]$ . Thus, if  $z \in C^0(\bar{B})$  solves  $P(f, R_1, H)$  where  $f \in C^0(\partial B)$  and  $\sup_{w \in \partial B} |f(w)| \leq r_1$ , then  $\tilde{z} \in \mathcal{D}_{r_1}$ .*

*Proof.* We first estimate the effect of  $T_*$  on the length of vectors. For  $V \in M_p$  we define an orthogonal decomposition  $V = V^r + V^s$  where  $V^r = \langle V, \gamma_* \rangle \gamma_*$ ,  $\gamma =$  arc-length geodesic from  $m$  to  $p$ . We have an analogous decomposition for  $\tilde{V} \in (M_m)_{\tilde{p}}$ , with  $V^r = \exp_*(\tilde{V}^r)$  and  $V^s = \exp_*(\tilde{V}^s)$ . Writing  $R = |\tilde{p}|$  we see that  $(\tilde{T}_* \tilde{V})^s = \tilde{V}^s \sigma(R)/R$  and  $(\tilde{T}_* \tilde{V})^r = \sigma'(R)\tilde{V}^r$ , modulo the identification of tangents to  $M_m$  at different points by parallel translation. Write  $\tilde{V}(\rho)$  for the Jacobi field along the ray through the origin and  $\tilde{p}$ , determined by  $\tilde{V}(R) = \tilde{V}$  and  $\tilde{V}(0) = 0$ . Namely  $\tilde{V}(\rho) = (\rho/R)\tilde{V}$  translated to  $(\rho/R)\tilde{p}$ . Let  $f(\rho)$  be the Jacobian of  $\exp$  restricted to the subspace generated by  $\tilde{V}(\rho)^s$ . For  $\rho_1 \leq \rho_2$  we have the inequality

$$\frac{\rho_1 f(\rho_1)}{\rho_2 f(\rho_2)} \leq \frac{\sin(b\rho_1)}{\sin(b\rho_2)},$$

(cf. [6] or [17, proof of Theorem 3]). This now yields:

$$\begin{aligned} |T_* V|^2 &= |(T_* V)^r|^2 + |(T_* V)^s|^2 = |(\tilde{T}_* \tilde{V})^r|^2 + f(\sigma(R))^2 |(\tilde{T}_* \tilde{V})^s|^2 \\ &= (\sigma'(R))^2 |\tilde{V}^r|^2 + (\sigma(R)/R)^2 f(\sigma(R))^2 |\tilde{V}^s|^2 \\ &= (\sigma'(R))^2 |V^r|^2 + \left( \frac{\sigma(R)f(\sigma(R))}{Rf(R)} \right)^2 |V^s|^2 \\ &\leq (\sigma'(R))^2 |V^r|^2 + \frac{\sin^2(b\sigma(R))}{\sin^2(bR)} |V^s|^2. \end{aligned}$$

Now the function  $\phi(R) = \sin(b\sigma(R))/\sin(bR)$  has  $\phi(r_1) = 1$  and  $\phi'(r_1) = 0$ . Since  $\sigma'(r_1) = 1$  and  $\sigma''(r_1+) < 0$ , we see that  $0 < \sigma' \leq \phi$  on some interval  $[0, R_0]$  where  $R_0 > r_1$ , equality holding on  $[0, r_1]$ . Then for  $R \leq R_0$

$$(4.1) \quad |T_* V|^2 \leq (\phi(R))^2 (|V^r|^2 + |V^s|^2) = \frac{\sin^2(b\sigma(R))}{\sin^2(bR)} |V|^2.$$

We need next to estimate the effect of  $T$  on the volume integrand  $\omega(z, \nabla z)$ .

Denote  $y = T \circ z$ , and let  $z_u^s(\rho)$ ,  $z_v^s(\rho)$  be the Jacobi fields generated by  $z_u^s$ ,  $z_v^s$ . Thus  $z_u^s(\rho) \in M_{\gamma(\rho)}$ . Observe that  $y_u^s(\rho) = z_u^s(\rho)$ ,  $y_v^s(\rho) = z_v^s(\rho)$  for  $\rho \leq \sigma(|\tilde{z}|)$ . First assume that  $z_u$  and  $z_v$  are independent, and let  $F(\rho)$  be the Jacobian of  $\exp$  at  $\rho\tilde{z}/R$ . Then for  $\rho_1 \leq \rho_2$

$$\frac{\rho_1^2 F(\rho_1)}{\rho_2^2 F(\rho_2)} \leq \frac{\sin^2(b\rho_1)}{\sin^2(b\rho_2)}$$

[6]. Now

$$\begin{aligned} \langle \gamma_*(\rho), *(y_u(\rho) \wedge y_v(\rho)) \rangle &= \langle \gamma_*(\rho), *(z_u(\rho) \wedge z_v(\rho)) \rangle \\ &= F(\rho) \frac{\tilde{z}}{R} \cdot \left( \frac{\rho}{R} \tilde{z}_u \right) \wedge \left( \frac{\rho}{R} \tilde{z}_v \right) C \rho^2 F(\rho), \end{aligned}$$

where  $C$  is independent of  $\rho$ . Thus using (2.1),

$$\begin{aligned} |\omega(z, \nabla z) - \omega(z, \nabla y)| &= \left| \int_{\sigma(R)}^R H \left( \frac{\rho z}{R} \right) C \rho^2 F(\rho) d\rho \right| \\ &\leq h |CR^2 F(R)| \int_{\sigma(R)}^R \frac{\rho^2 F(\rho)}{R^2 F(R)} d\rho \\ &\leq h |\langle \gamma_*(R), *(z_u \wedge z_v) \rangle| \int_{\sigma(R)}^R \frac{\sin^2(b\rho)}{\sin^2(bR)} d\rho \\ &\leq \frac{1}{2} h |\nabla z|_M^2 \int_{\sigma(R)}^R \frac{\sin^2(b\rho)}{\sin^2(bR)} d\rho. \end{aligned}$$

If  $z_u$  and  $z_v$  are not independent, this relation holds trivially.

Finally, with  $i(z, \nabla z) = |\nabla z|_M^2 + 4\omega(z, \nabla z)$ , and supposing  $R = |\tilde{z}| \leq R_0$  we have from (4.1) and (4.2) that

$$\begin{aligned} i(z, \nabla z) - i(y, \nabla y) &= |\nabla z|_M^2 - |\nabla y|_M^2 + 4(\omega(z, \nabla z) - \omega(y, \nabla y)) \\ &\geq |\nabla z|_M^2 \left\{ 1 - \frac{\sin^2(b\sigma(R))}{\sin^2(bR)} - 2h \int_{\sigma(R)}^R \frac{\sin^2(br)}{\sin^2(bR)} d\rho \right\} \\ &= |\nabla z|_M^2 g(R). \end{aligned}$$

In straightforward fashion we compute  $g(R) = 0$  for  $R \leq r_1$ ,  $g'(r_1) = 0$ , and

$$g''(r_1+) = 2\sigma''(r_1+)[h - b \cot(br_1)] > 0.$$

Thus there exists  $R_1 \in (r_1, R]$  such that  $g > 0$  on  $(r_1, R_1]$ . Now assume  $z \in \mathcal{D}_{R_1}$ . We have  $i(z, \nabla z) - i(y, \nabla y) \geq 0$  everywhere, i.e.,  $E_{B''}[z] - E_{B''}[y] \geq 0$  for all measurable  $B'' \subset B$ . Assume further  $z \in C^0(\bar{B})$  with



$$\inf_{w \in B} |\tilde{z}(w)| \leq r_1 < R_2 = \sup_{w \in B} |\tilde{z}(w)| \leq R_1 .$$

Then  $|\tilde{z}|$  takes every value in  $[r_1, R_2]$ . In particular,  $z$  is not constant on the open set

$$B' = \{w \in B : \frac{1}{2}(r_1 + R_2) < |\tilde{z}(w)| < R_2\} ,$$

and thus  $D_{B'}[z] > 0$ . But there exists  $\delta > 0$  such that  $g \geq \delta$  on  $[\frac{1}{2}(r_1 + R_2), R_2]$ . Let  $B'' = B \setminus B'$ . Hence

$$E[z] - E[y] = E_{B''}[z] - E_{B''}[y] + E_{B'}[z] - E_{B'}[y] \geq \delta D_{B'}[z] > 0 ,$$

as claimed.

Now, if  $\sup_{w \in \partial B} |f(w)| \leq r_1$  and  $z \in \mathcal{D}_{R_1}(f)$  then  $y \in \mathcal{D}_{R_1}(f)$ ; thus if  $z$  solves  $P(f, R_1, H)$ ,  $\sup_{w \in B} |\tilde{z}(w)| > r_1$  is impossible.

**Theorem 1.** *Suppose  $M^3$  is a riemannian manifold of class  $C^3$  with sectional curvatures  $\leq b^2$ . For some  $m \in M$  and  $r_0 > 0$ , with  $4b^2r_0^2 < \pi^2$ , suppose that the restriction of  $\exp = \exp_m$  to  $B_{r_0}(0) \subset M_m$  is a diffeomorphism onto its image. Let a  $C^1$  function  $H: B_{r_0} \rightarrow \mathbf{R}$  be given with  $h = \sup_{x \in \bar{B}_{r_0}} |H(x)| < b \cot(br_1)$ , where  $r_1 < r_0$ . Then given  $f \in \mathcal{D}_{r_1} \cap C^0(\partial B)$  there exists a solution  $z \in C^2(B) \cap C^0(\bar{B})$  to  $P(f, r_1, H)$ , which satisfies (1.1) in  $B$  and agrees with  $\exp \circ f$  on  $\partial B$ .*

*Proof.* Let  $R_1$  be as given by Lemma 5. Let  $z$  be a solution to  $P(f, R_1, H)$  as given by Lemma 4. From results of Morrey we have  $z \in C^0(\bar{B})$ , as remarked at the end of § 3, and  $z \in C^2(B')$  for any  $B' \subset B$  with  $\sup_{w \in B'} |\tilde{z}(w)| < R_1$ . But Lemma 5 shows that  $\sup_{w \in B} |\tilde{z}(w)| \leq r_1 < R_1$ , so that  $z \in C^2(B)$  and satisfies the Euler equations (1.1). q.e.d.

We now drop the condition that  $H \in C^1$ . Given any function  $H \in C^a(\bar{B}_{r_1})$  with  $\sup_{x \in \bar{B}_{r_1}} |H(x)| \leq b \cot(br_1)$  we approximate  $H$  uniformly in  $B_{r_1}$  by a sequence of functions  $H_n \in C^1(B_{r_0})$  with  $\sup_{x \in \bar{B}_{r_0}} |H_n(x)| < b \cot(br_1)$ . Then for each  $H_n$  Theorem 1 gives a solution  $z_n$  to  $P(f, r_1, H_n)$ . As in [5, § 5] we find  $z$  such that some subsequence of the  $z_n$  converges in  $H_1(B)$  to  $z$ . Since each  $z_n$  has a representation (3.1) with respect to  $H_n$ , we obtain that representation for  $z$  with respect to  $H$ . It may then be shown, using a standard argument of potential theory, that  $z \in C^{2+\alpha}$  and satisfies (1.1). This shows

**Corollary 2.** *Theorem 1 continues to hold if the function  $H$  satisfies only  $H \in C^a(\bar{B}_{r_1})$  and  $\sup_{x \in \bar{B}_{r_1}} |H(x)| \leq b \cot(br_1)$ .*

### 5. The Plateau problem

Let  $\Gamma$  be an oriented closed Jordan curve in  $B_{r_0}$ . Denote by  $\mathcal{D}(\Gamma, R)$  the

set of mappings  $\tilde{x} \in \mathcal{D}_R$  such that  $x|_{\partial B}$  is equal almost everywhere to a continuous, monotone mapping of degree 1 over the integers onto  $\Gamma$ . Define a variational problem  $P_H(\Gamma, R)$  by  $E[x] \rightarrow \min$  among mappings  $x$  such that  $\tilde{x} \in \mathcal{D}(\Gamma, R)$ .

**Theorem 2.** *Let  $M$  be a riemannian manifold of dimension 3 and of class  $C^3$  with sectional curvatures  $\leq b^2$ . For  $m \in M$  and  $r_1 > 0$  with  $4r_1^2b^2 < \pi^2$ , assume  $\exp_m$  is defined on  $\bar{B}_{r_1}(0) \subset M_m$  and maps  $\bar{B}_{r_1}(0)$  diffeomorphically onto  $\bar{B}_{r_1} = \bar{B}_{r_1}(m)$ . Let  $\Gamma$  be an oriented closed Jordan curve in  $\bar{B}_{r_1}$  such that  $\mathcal{D}(\Gamma, \infty)$  is nonempty. Let  $H$  be a uniformly Hölder-continuous function:  $\bar{B}_{r_1} \rightarrow \mathbb{R}$  with  $h = \sup_{z \in \bar{B}_{r_1}} |H(z)| \leq b \cot(br_1)$ . Then there exists a solution  $z \in C^2(B) \cap C^0(\bar{B})$  to the variational problem  $P_H(\Gamma, r_1)$ , mapping  $\partial B$  homeomorphically onto  $\Gamma$  in an orientation-preserving fashion and satisfying (1.1) and (1.2) in  $B$ .*

*Proof.* Let  $r_0 > r_1$  be chosen so that  $4b^2r_0^2 < \pi^2$  and so that  $\exp_m$  is a diffeomorphism of  $B_{r_0}(0)$  onto  $B_{r_0}(m)$ . The theorem now follows from Corollary 2 in essentially the same fashion as in [11, Theorem 2]. In the process of the proof, we modify a minimizing sequence  $\{x_n\}$  by requiring each  $x_n$  to satisfy a three-point condition. This can be done without changing  $E[x_n]$  since  $E$  is conformally invariant. We require the choice of the three points to be such that every monotone map:  $\partial B \rightarrow \Gamma$  satisfying the three-point condition will be of degree 1. In particular, the limiting mapping  $z|_{\partial B}$  will be of degree 1. Observe that for any  $C^1$ -diffeomorphism  $\phi: \bar{B} \rightarrow \bar{B}$  there holds  $W[\phi \circ z] = W[z]$ ; therefore since  $\phi \circ z \in \mathcal{D}(\Gamma, r_1)$  we have  $D[z] \leq D[\phi \circ z]$ . From this it follows that  $z$  satisfies (1.2) by a straightforward adaptation of the method of [1, pp. 107-112].

To show that  $z|_{\partial B}$  is a homeomorphism, it suffices to show that for any  $w_0 \in \partial B$ , a neighborhood of which in  $\partial B$  is mapped into a  $C^2$  curve, there holds an asymptotic representation

$$z_u - iz_v = a(w - w_0)^l + O(|w - w_0|^l)$$

for some integer  $l \geq 1$  and  $a \in C^0 \setminus \{0\}$ . This may be obtained by suitable modification of an argument of Heinz [9, relations (14) and (30)] to allow isothermal parameters in the sense of (1.2).

## 6. Globalization

We now drop the requirement that  $\exp$  be injective on  $\bar{B}_{r_1}(0)$ . We shall need the following fact, which may be expressed as the statement that  $\exp$  behaves like a covering projection with respect to curves which are not too long.

**Lemma 6.** *Let  $M$  be a complete riemannian manifold of class  $C^3$  with sectional curvatures  $\leq b^2$ . Suppose a  $C^1$  curve  $\gamma: [0, 1] \rightarrow M$  is given with  $\gamma(0) = m$  and  $r = \text{length}(\gamma) < r_1$ , where  $r_1^2b^2 < \pi^2$ . Then there is a unique mapping*

$\tilde{\gamma}: [0, 1] \rightarrow M_m$  with  $\tilde{\gamma}(0) = 0$  and  $\exp \circ \tilde{\gamma} = \gamma$ . Moreover, suppose  $\{\gamma_s\}$  is a family of such curves such that  $g(s, t) = \gamma_s(t)$  defines a continuous mapping  $g: (-\varepsilon, \varepsilon) \times [0, 1] \rightarrow M$ . Then the family  $\{\tilde{\gamma}_s\}$  of liftings yields a continuous mapping  $\tilde{g}: (-\varepsilon, \varepsilon) \times [0, 1] \rightarrow M_m$  by defining  $\tilde{g}(s, t) = \tilde{\gamma}_s(t)$ .

*Proof.* Every point  $q \in B_{r_1}(0) \subset M_m$  has a neighborhood  $U(q)$  such that  $\exp$  is a diffeomorphism of  $U(q)$  onto its image. This follows from the condition  $r_1^2 b^2 < \pi^2$  using a comparison technique (cf. [3, pp. 176–179]). Let  $S$  be the set of  $t \in [0, 1]$  such that there exists a unique continuous lifting  $\tilde{\gamma}: [0, t] \rightarrow M_m$  with  $\exp \circ \tilde{\gamma} = \gamma|_{[0, t]}$  and  $\tilde{\gamma}(0) = 0$ . Thus  $0 \in S$ . Suppose  $t \in S$ . Then for  $t_1 \leq t$ ,

$$|\tilde{\gamma}(t_1)| = \int_0^{t_1} \tilde{\gamma}_*(t) \cdot \tilde{W} dt = \int_0^{t_1} \langle \gamma_*(t), W \rangle dt \leq \text{length}(\gamma) = r,$$

where  $\tilde{W}$  is the radial unit vector field in  $M_m$ , and  $W = \exp_*(\tilde{W})$ . Thus  $\tilde{\gamma}([0, t]) \subset B_r(0) \subset \subset B_{r_1}(0)$ .

In particular, for sufficiently small  $\varepsilon > 0$ ,  $\gamma([t, t + \varepsilon]) \subset \exp(U(\tilde{\gamma}(t)))$  so that for  $t_1 \in [t, t + \varepsilon]$  defining  $\tilde{\gamma}(t_1) = (\exp|_{U(\tilde{\gamma}(t))})^{-1} \circ \gamma(t_1)$  extends  $\tilde{\gamma}$  over  $[0, t + \varepsilon]$ . The extended curve  $\tilde{\gamma}$  must be unique, since otherwise this process would provide a contradiction to the uniqueness of  $\tilde{\gamma}|_{[0, t]}$ . Thus  $t + \varepsilon \in S$ . This shows  $S_0 = \{t_0: [0, t_0] \subset S\}$  is open.

Now suppose  $\{t_n\}$  is an increasing sequence from  $S$  with  $t_n \rightarrow t_0$ . Uniqueness implies that the curves  $\tilde{\gamma}$  associated with different values  $t_n$  are merely restrictions of one another. This defines  $\tilde{\gamma}: [0, t_0] \rightarrow B_r(0)$ . Among the various points of the finite set  $\{q \in \bar{B}_r(0): \exp(q) = \gamma(t_0)\}$ , at most one can be a cluster point of  $\tilde{\gamma}(t)$  as  $t \rightarrow t_0$ ; otherwise there would be a continuum of cluster points, each of which must be mapped to  $\gamma(t_0)$  by  $\exp$ . It follows that  $\tilde{\gamma}(t)$  approaches a limit as  $t \rightarrow t_0$ , and we define  $\tilde{\gamma}(t_0)$  to be this limit. The uniqueness of this extended  $\tilde{\gamma}$  is clear. Thus  $t_0 \in S$ . This shows  $S_0$  is closed. Therefore  $S_0 = [0, 1]$ , i.e.,  $S = [0, 1]$ .

For the second part of the conclusion, it suffices to show  $\tilde{g}$  is continuous at  $s = 0$ . The compact set  $\tilde{\gamma}([0, 1])$  is covered by a finite number of neighborhoods  $U_i, 1 \leq i \leq n$ , where each  $U_i$  is  $U(\tilde{\gamma}(t))$  for some  $t \in [0, 1]$ . Choose  $\delta \in (0, \varepsilon)$  small enough that for each  $t \in [0, 1], g([-\delta, \delta] \times \{t\}) \subset \exp(U_i)$  for some  $i$ . Define  $T = \{t \in [0, 1]: \tilde{g} \text{ is continuous on } [-\delta, \delta] \times [0, t]\}$ . Thus  $0 \in T$ . For some  $t \geq 0$ , suppose  $[0, t] \subset T$ . There exist  $i$  and  $\eta > 0$  such that  $g([-\delta, \delta] \times [t - \eta, t + \eta]) \subset \exp(U_i)$ . For  $s \in [-\delta, \delta]$  and  $t_1 \in [t - \eta, t + \eta]$  define  $\tilde{\gamma}'_s(t_1) = (\exp|_{U_i})^{-1} \circ \gamma_s(t_1)$ ; this defines a continuous lifting  $\tilde{\gamma}'_s$  of  $\gamma_s|_{[0, t + \eta]}$ . By the uniqueness of  $\tilde{\gamma}_s$ , we have  $\tilde{\gamma}'_s = \tilde{\gamma}_s$ , i.e.,

$$\tilde{g}|_{[-\delta, \delta] \times [t - \eta, t + \eta]} = (\exp|_{U_i})^{-1} \circ g|_{[-\delta, \delta] \times [t - \eta, t + \eta]}.$$

Thus  $\tilde{g}$  is continuous on  $[-\delta, \delta] \times [t - \eta, t + \eta]$  and hence on  $[-\delta, \delta] \times [0, t + \eta]$  via a glueing lemma. So  $t + \eta \in T$ . This shows  $T = [0, 1]$ , i.e.,  $\tilde{g}$  is continuous. q.e.d.

We shall need a new way of limiting the extent of a closed contractible curve  $\Gamma: [0, 1] \rightarrow M$ . Let a contraction of  $\Gamma$  be given by  $g: [0, 1] \times [0, 1] \rightarrow M$  with  $g(s, 0) = m, g(s, 1) = \Gamma(s)$  and  $g(1, t) = g(0, t)$  for all  $s, t \in [0, 1]$ . We may assume the transverse curves  $g_s(t) = g(s, t)$  are uniformly smooth:  $g_s \in C^1([0, 1])$  and  $\sup \text{length}(g_s) < \infty$ . We make the following definition: if  $g$  is a contraction of  $\Gamma$  such that each  $g_s$  is rectifiable and  $\text{length}(g_s) \leq r$ , we call  $g$  an  $r$ -contraction of  $\Gamma$ ; if  $\Gamma$  has an  $r$ -contraction, it is called  $r$ -contractible. Thus any contractible curve is  $r$ -contractible for sufficiently large  $r$ .

**Lemma 7.** *Let  $N$  be a complete riemannian manifold of class  $C^3$  with sectional curvatures  $\leq b^2$ . If a continuous closed curve  $\Gamma: [0, 1] \rightarrow N$  is  $r_1$ -contractible, where  $b^2 r_1^2 < \pi^2$ , then there exist  $n \in N$  and a continuous closed curve  $\tilde{\Gamma}: [0, 1] \rightarrow \bar{B}_{r_1}(0) \subset N_n$  such that  $\Gamma = \exp_n \circ \tilde{\Gamma}$ .*

*Proof.* Let  $g: [0, 1] \times [0, 1] \rightarrow N$  be an  $r_1$ -contraction of  $\Gamma$ , and  $n$  the common point  $g(s, 0)$ . Write  $g_s(t) = g(s, t)$ ; we have  $\text{length}(g_s) \leq r_1$ . Applying Lemma 6 to the family of curves  $\{g_s\}$ , there is a family of liftings  $\{\tilde{g}_s\}$  such that  $\tilde{g}(s, t) = \tilde{g}_s(t)$  defines a continuous mapping  $\tilde{g}: [0, 1] \times [0, 1] \rightarrow N^n$ . Since  $g_0 = g_1$ , it follows from the uniqueness of liftings that  $\tilde{g}_1 = \tilde{g}_0$ . Let  $\tilde{\Gamma}(s) = \tilde{g}_s(1)$ . Then  $\tilde{\Gamma}$  is a continuous closed curve with  $\Gamma = \exp_n \circ \tilde{\Gamma}$ .

**Theorem 3.** *Let  $\Gamma$  be an  $r_1$ -contractible Jordan curve in a complete-riemannian manifold  $N^3$  of class  $C^3$  and with sectional curvatures  $\leq b^2$ . Assume there is a mapping  $x_0: \bar{B} \rightarrow N$  such that  $x_0$  maps  $\partial B$  continuously and monotonically onto  $\Gamma$ , and  $D[x_0] < \infty$ . Suppose that  $H \in C^\alpha(N)$  satisfies  $\sup_{x \in N} |H(x)| \leq b \cot(br)$  and that  $4b^2 r_1^2 < \pi^2$ . Then there is a mapping  $z: \bar{B} \rightarrow N, z \in C^2(B) \cap C^0(\bar{B})$ , taking  $\partial B$  homeomorphically onto  $\Gamma$  and satisfying (1.1) and (1.2) in B.*

*Proof.* By Lemma 7, there exist  $n \in N$  and a continuous closed curve  $\tilde{\Gamma}: [0, 1] \rightarrow \bar{B}_{r_1}(0) \subset N_n$  such that  $\Gamma = \exp_n \circ \tilde{\Gamma}$ . Thus  $\tilde{\Gamma}$  is a Jordan curve. We shall define a new manifold  $M$  as follows. Let  $r_0 > r_1$  be chosen with  $b^2 r_0^2 < \pi^2$ . Then  $\exp_n$  has full rank on  $B_{r_0}(0) \subset N_n$ . Let  $M$  be  $B_{r_0}(0)$  with the riemannian structure which makes  $\exp_n$  a local isometry, and denote  $m = 0 \in N_n$ . Then clearly  $\exp_m$  is a diffeomorphism of  $B_{r_0}(0) \subset M_m$  onto  $M = B_{r_0}(m)$ . Define  $\tilde{H}: M \rightarrow R$  by  $\tilde{H}(x) = H \circ \exp_n(x)$ . We need to find  $y_0 \in \mathcal{D}(\tilde{\Gamma}, \infty)$ . We may assume  $x_0$  is smooth in  $B$ . It is then possible to modify  $x_0$  on some compact subdomain of  $B$  to a mapping  $x_1$  which is smooth in  $B$  and describes an  $r_0$ -contraction of  $\Gamma$  to  $n$  such that  $x_1$  is homotopic through  $r_0$ -contractions of  $\Gamma$  to  $\exp_n(C(\tilde{\Gamma}))$ . Lemma 6 may then be applied to find a lifting  $y_0: \bar{B} \rightarrow N_n$  with  $x_1 = \exp_n \circ y_0$ . Thus  $y_0 \in \mathcal{D}(\tilde{\Gamma}, \infty)$ . Now apply Theorem 2 to the curve  $\tilde{\Gamma}$  in the manifold  $M$  with the prescribed function  $\tilde{H}$ : this gives a mapping  $y: \bar{B} \rightarrow M$ . Define  $z = \exp_n \circ y$ . Then  $z$  has the required properties.

**Remarks.** 1) It is clear from the proof that weaker hypotheses will suffice: if  $\Gamma$  is  $r_1$ -contractible to a point  $n \in N$ , then we may replace the requirement that  $N$  be complete by the requirement that  $\exp_n$  be defined on  $\bar{B}_{r_1}(0) \subset N_n$ , and require  $H$  to be defined only on  $\bar{B}_{r_1}(n)$ , of class  $C^\alpha(\bar{B}_{r_1}(n))$ , with

$$\sup_{x \in \bar{B}_{r_1}(\pi)} |H(x)| \leq b \cot(br_1).$$

2) Observe that the solution mapping is homotopic to the particular  $r_1$ -contraction  $g$  employed in the proof, up to sign; that is, the two mappings represent the same element or inverse elements in  $\pi_2(N, \Gamma)$ . In fact either orientation could be specified for  $\Gamma$  so that  $z \in [g]$  or  $z \in [g]^{-1}$  could be obtained at will. Thus, we may obtain a solution  $z$  in any homotopy class in which  $\Gamma$  is  $r_1$ -contractible.

3) The author [4] has recently demonstrated that the solution mapping  $z$  is an immersion, that is,  $\langle z_u, z_u \rangle = \langle z_v, z_v \rangle \neq 0$  in  $B$ .

4) By a result of Heinz [8], the restriction on  $h$  is the best possible for the case  $b = 0$ ; it is reasonable to suppose that it continues to be sharp for other values of  $b$ .

5) The requirement that  $\Gamma$  be  $r_1$ -contractible may not be replaced by the condition of contractibility in conjunction with a general restriction on diameter. This may be seen by considering a flat three-torus  $T^3$  of arbitrarily small diameter, letting  $\Gamma$  be the image of a plane circle of radius  $> h^{-1}$  under the locally isometric covering map  $E^3 \rightarrow T^3$ . Using the result of [8], this problem has no solution with  $H(x) \equiv h$ .

### 7. Minimal surfaces

In the case  $H \equiv 0$ , we may ignore the volume term  $W[z]$  entirely, and the restriction on the dimension of  $M$  is no longer necessary. The same considerations, with inessential modifications, now yield:

**Theorem 4.** *Let  $\Gamma$  be an  $r_1$ -contractible Jordan curve in a complete riemannian manifold  $N$ , of class  $C^3$  and with sectional curvatures  $K \leq K_0$ . Assume there is a mapping  $x_0: \bar{B} \rightarrow N$  which takes  $\partial B$  continuously and monotonically onto  $\Gamma$ , with  $D[x_0] < \infty$ . Suppose  $4K_0r_1^2 < \pi^2$ . Then there is a minimal surface in  $N$  with a conformal representation  $z \in C^2(B) \cap C^0(\bar{B})$  mapping  $\partial B$  homeomorphically onto  $\Gamma$ .*

This is a partial generalization of the theorem of Morrey [14].

**Remarks.** 1) If  $\dim N > 3$ , we make no claim that the solution mapping will be an immersion.

2) In [14], Morrey constructs an example to shed light on his hypothesis of homogeneous regularity. The example occurs in a manifold of negative sectional curvature; but in such a manifold Theorem 4 gives a minimal surface spanning every contractible rectifiable Jordan curve. Thus no example has yet come to light of a contractible rectifiable Jordan curve in a complete manifold which cannot be spanned by a minimal surface.

### Bibliography

- [ 1 ] R. Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces*, Interscience, New York, 1950.
- [ 2 ] J. Douglas, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. **33** (1931) 263-321.
- [ 3 ] D. Gromoll, W. Klingenberg & W. Meyer, *Riemannsche Geometrie im Grossen*, Lecture Notes in Math. Vol. 55, Springer, Berlin, 1968.
- [ 4 ] R. Gulliver, *Regularity of minimizing surfaces of prescribed mean curvature*, Ann. of Math. **97** (1973) 275-305.
- [ 5 ] R. Gulliver & J. Spruck, *The Plateau problem for surfaces of prescribed mean curvature in a cylinder*, Invent. Math. **13** (1971) 169-178.
- [ 6 ] P. Günther, *Einige Sätze über das Volumenelement eines Riemannschen Raumes*, Publ. Math. Debrecen **7** (1960) 78-93.
- [ 7 ] E. Heinz, *Über die Existenz einer Fläche konstanter mittlerer Krümmung bei vorgegebener Berandung*, Math. Ann. **127** (1954) 258-287.
- [ 8 ] —, *On the non-existence of a surface of constant mean curvature with finite area and prescribed rectifiable boundary*, Arch. Rational Mech. Anal. **35** (1969) 249-252.
- [ 9 ] —, *Über das Randverhalten quasilinearer elliptischer Systeme mit isothermen Parametern*, Math. Z. **113** (1970) 99-105.
- [ 10 ] S. Hildebrandt, *Randwertprobleme für Flächen mit vorgeschriebener mittlerer Krümmung und Anwendungen auf die Kapillaritätstheorie. I*, Math. Z. **112** (1969) 205-213.
- [ 11 ] —, *On the Plateau problem for surfaces of constant mean curvature*, Comm. Pure Appl. Math. **23** (1970) 97-114.
- [ 12 ] H. Kaul, *Ein Einschliessungssatz für H-Flächen in Riemannschen Mannigfaltigkeiten*, Manuscripta Math. **5** (1971) 103-112.
- [ 13 ] C. B. Morrey, Jr., *Multiple integral problems in the calculus of variations and related topics*, Univ. California Publ. Math., New series, Vol. 1, No. 1, 1943, 1-130.
- [ 14 ] —, *The problem of Plateau on a Riemannian manifold*, Ann. of Math. **49** (1948) 807-851.
- [ 15 ] —, *Multiple integrals in the calculus of variations*, Springer, New York, 1966.
- [ 16 ] T. Radó, *The problem of the least area and the problem of Plateau*, Math. Z. **32** (1930) 763-796.
- [ 17 ] H. E. Rauch, *A contribution to Riemannian geometry in the large*, Ann. of Math. **54** (1951) 38-55.
- [ 18 ] F. Tomi, *Ein einfacher Beweis eines Regularitätssatzes für schwache Lösungen gewisser elliptischer Systeme*, Math. Z. **112** (1969) 214-218.
- [ 19 ] H. Werner, *Das Problem von Douglas für Flächen konstanter mittlerer Krümmung*, Math. Ann. **133** (1957) 303-319.

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